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LARGE AMPLITUDE RADIAL OSCILLATIONS OF LAYERED THICK-WALLED CYLINDRICAL SHELLS

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Abstract—Finite breathing motions of multi-layered, long, circular cylindrical shells of arbitrary wall thickness are investigated on the basis of the theory of large elastic deformations. The materials of the layers are assumed to be isotropic, elastic, homogeneous and incompressible. The governing non-linear ordinary differential equation is solved partially to give the frequencies of oscillations in an integral form. An approximate solution technique based on Galerkin's orthogonalization process is also formulated to give complete solutions. A tube consisting of two layers of neo-Hookean materials is solved both by exact and approximate methods. An excellent agreement is observed between the two sets of results.

INTRODUCTION

The dynamic behavior of thick-walled bodies undergoing large elastic deformations has attracted attentions of several researchers after the notable works of Knowles[1, 2] on the large amplitude radial oscillations of long circular cylindrical tubes. Based on Knowles work, Guo and Solecki[3] and Wang[4] analyzed finite amplitude oscillations of spherical shells. In all these works, the material of the shell is assumed to be elastic, homogeneous, isotropic and incompressible. The incompressibility condition reduces the problems to a single degree of freedom system and, as a result, the non-linear equation of motion becomes integrable. When oscillatory motions exist, an expression for the frequency of oscillations is obtained in a form involving an improper integral.

Nowinski and Wang[5] developed a Galerkin type process for thick-walled cylinders to obtain an approximate but complete solution to the problem. Wang and Ertepinar[6] adopted the Galerkin procedure stated in[5] to study the large amplitude oscillations of laminated thick-walled spherical shells. The results obtained in[6] compared favorably with those obtained by the exact formulation for moderately large amplitudes. Recently, Benveniste[7] considered the same problem and solved the governing non-linear ordinary differential equation by first decomposing it into two first-order equations and then applying a Runge–Kutta integration scheme. Numerical results are given in[7] for three, four and five layer shells.

In the present work, large amplitude radial oscillations of multi-layered, long, circular, cylindrical tubes are investigated. The layers are assumed to be of arbitrary thicknesses and made of isotropic, elastic, homogeneous and incompressible materials. The formulation is based on the theory of finite elastic deformations[8]. The tube is assumed to undergo radial motions by a sudden application of inside gas pressure. The pressure is assumed to obey ideal gas law. The governing non-linear ordinary differential equation is written in an integrated form suitable to obtain an expression for the frequencies of oscillations. To provide some numerical results, a tube of two layers, each made of a different neo-Hookean material, is considered. An approximate solution procedure suggested in[5] is also formulated and applied to a tube of identical properties. An excellent agreement is observed between the two sets of results for all amplitudes.

FORMULATION OF THE PROBLEM

Consider a long, circular, cylindrical tube consisting of N concentric layers of arbitrary thicknesses. The material of a layer is assumed to be isotropic, elastic, homogeneous and

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incompressible with a strain energy density function $W_i(I, II)$ where i denotes the layer and I and II are, respectively, the first and the second strain invariants. If the tube undergoes pure radial motions (breathing motions) due to a sudden pressure pulse applied uniformly over the inner lateral surface of the tube at $t=0$, a material point at (r, θ, z) in the current state at time t is at (R, θ, z) in the unstressed state. Due to the incompressibility of the layers

$$R^2 - r^2 = R_i^2 - r_i^2, \quad i = 1, 2, \ldots, N \tag{1}$$

where R_i and r_i denote, respectively, the inner radii of the ith layer at the unstressed and stressed states. Equation (1) implies that the radial motion of the tube is completely described once the inner radius r_1 is determined as a function of time. Letting $i = 1$, and differentiating with respect to time twice, eqn (1) gives

$$r\ddot{r} = r_1\ddot{r}_1 - \frac{1}{r^2} r_1^2 \dot{r}_1^2 \tag{2}$$

where a dot denotes partial differentiation with respect to time.

Introducing the notation

$$Q = r^2/R^2, \quad Q_i = r_i^2/R_i^2, \quad i = 1, 2, \ldots, N \tag{3}$$

the strain invariants and the stress field are given by

$$I = II = 1 + Q + Q^{-1}, III = 1, \tag{4}$$

$$\begin{aligned} \tau^{11} &= \Phi Q^{-1} + \psi(1 + Q^{-1}) + p, \\ \tau^{22} &= r^{-2}[\Phi Q + \psi(1 + Q) + p], \\ \tau^{33} &= \Phi + \psi(Q + Q^{-1}) + p, \\ \tau^{12} &= \tau^{23} = \tau^{31} = 0, \end{aligned} \tag{5}$$

where

$$\Phi = 2\frac{\partial W}{\partial I}, \quad \psi = 2\frac{\partial W}{\partial II}, \tag{6}$$

and p is an unknown pressure.

The equations of motion in θ and z directions imply that p is a function of r and t only. The motion of the tube is governed by the equation of motion in the radial direction,

$$\frac{\partial \tau^{11}}{\partial Q} = \frac{1}{(1-Q)}\frac{\partial W}{\partial Q} + \frac{\rho}{2}\left[\frac{1}{Q(1-Q)}(r_1\ddot{r}_1 + \dot{r}_1^2) + \frac{1}{Q^2}\frac{r_1^2\dot{r}_1^2}{R_1^2(Q_1 - 1)}\right] \tag{7}$$

where ρ denotes the current mass density at r.

Integrating eqn (7) from Q_1 to Q_{N+1} and considering continuity conditions at the interfaces

$$\bar{p} = f(W_i) + (\eta\ddot{\eta} + \dot{\eta}^2)\ln\left[\frac{(\eta^2 + S_1)^{m_1}(\eta^2 + S_2)^{m_2}\ldots(\eta^2 + S_N)^{m_N}}{(\eta^2 + S_0)^{m_1}(\eta^2 + S_1)^{m_2}\ldots(\eta^2 + S_{N-1})^{m_{N-1}}}\right]$$
$$- \sum_{i=1}^{N} m_i\eta^2\dot{\eta}^2\frac{(S_i - S_{i-1})}{(\eta^2 + S_i)(\eta^2 + S_{i-1})} \tag{8}$$

where

$$\begin{aligned} &\eta = r_1/R_1, \quad S_i = \frac{R_{i+1}^2}{R_i^2} - 1, \\ &m_i = \rho_i/\rho_0, \quad \rho_0 = \text{a reference density}, \\ &\bar{p} = \frac{2p_{in}}{\rho_0 R_1^2}, \quad p_{in} = \text{current inside pressure}, \\ &f(W_i) = -\frac{1}{\rho_0 R_1^2}\sum_{i=1}^{N}\int_{Q_i}^{Q_{i+1}}(\phi_i + \psi_i)\frac{(1+Q)}{Q^2}\,\mathrm{d}Q. \end{aligned} \tag{9}$$

Here, we note that, eqn (8) reduces to eqn (3) of Ref. [2] for a single layer tube.

Equation (8) is the exact second order non-linear differential equation governing the motion. It can be rewritten in an integrated form as

$$\eta\bar{p} = \eta f(W_i) + \frac{d}{d\eta}\left[\frac{1}{2}\eta^2\dot{\eta}^2\sum_{i=1}^{N}\ln\frac{(\eta^2+S_i)^{m_i}}{(\eta^2+S_{i-1})^{m_i}}\right]. \quad (10)$$

APPLICATION TO A TWO-LAYER TUBE

Let the inside gas pressure change according to the ideal gas law, pv^α = constant, where α is the polytropic constant and v is the specific volume. Thus, if p_{in}^0 is the initial pressure, then

$$\bar{p} = \frac{2p_{in}^0\eta^{-2\alpha}}{\rho_0 R_1^2}. \quad (11)$$

Assume that the layers are made of neo-Hookean materials with strain energy density functions

$$W_1 = C_1(I-3) = \frac{\phi_1}{2}(I-3), \quad W_2 = C_1'(I-3) = \frac{\phi_2}{2}(I-3). \quad (12)$$

The frequency of small amplitude oscillations is obtained by letting $\eta = 1+\epsilon$, $\epsilon \ll 1$, in eqn (8) and then solving the resulting second order, linear, ordinary differential equation:

$$\omega_0^2 = \frac{4}{\rho_0 R_1^2}\frac{\left[\dfrac{\phi_1 S_1}{(1+S_1)} + \dfrac{\phi_2(S_2-S_1)}{(1+S_1)(1+S_2)}\right]}{\ln\left[\dfrac{(1+S_1)^{m_1}(1+S_2)^{m_2}}{(1+S_1)^{m_2}}\right]}, \quad (13)$$

which reduces to

$$\omega_0^2 = \frac{2\Phi}{\rho}\frac{R_2^2 - R_1^2}{R_1^2 R_2^2 \ln\left(\dfrac{R_2}{R_1}\right)} \quad (14)$$

for a single layer tube. Equation (14) compares with that obtained in Ref. [2].

To solve the problem of large amplitude oscillations partially, eqns (11) and (12) are substituted into eqn (10) and the resulting equation is integrated subject to initial conditions $\eta(0) = 1$, $\dot{\eta}(0) = 0$,

$$\dot{\eta}^2 = \frac{2}{\eta^2 \ln\left[\dfrac{(\eta^2+S_1)^{m_1}(\eta^2+S_2)^{m_2}}{\eta^{2m_1}(\eta^2+S_1)^{m_2}}\right]}\left[\int_1^\eta \eta\bar{p}\,d\eta - \int_1^\eta \eta f_1(W)\,d\eta - \int_1^\eta \eta f_2(W)\,d\eta\right], \quad (15)$$

where

$$\int_1^\eta \eta\bar{p}\,d\eta = \begin{cases} \dfrac{2p_{in}^0}{\rho_0 R_1^2}\cdot\dfrac{(\eta^{2(1-\alpha)}-1)}{(1-\alpha)} & \text{for } \alpha \neq 1, \\ \dfrac{2p_{in}^0}{\rho_0 R_1^2}\ln(\eta)^2 & \text{for } \alpha = 1, \end{cases}$$

$$\int_1^\eta \eta f_1(W)\,d\eta = \frac{\phi_1}{\rho_0 R_1^2}(1-\eta^2)\ln\frac{(\eta^2+S_1)}{\eta^2(1+S_1)}, \quad (16)$$

$$\int_1^\eta \eta f_2(W)\,d\eta = \frac{\phi_2}{\rho_0 R_1^2}(1-\eta^2)\ln\frac{(1+S_1)(\eta^2+S_2)}{(1+S_2)(\eta^2+S_1)}.$$

The frequency of oscillations is determined from

$$\omega = \pi \Big/ \int_1^{\eta^+} \eta^{-1}\,d\eta \quad (17)$$

where $\eta^*(\eta^* > 1)$ is the other root of eqn (15). The limitations on the strain energy density functions for eqn (15) to possess a real root $\eta^* > 1$ have been studied in detail in [2, 6]. Applying the same argument, it is assumed that W_1 and W_2 are monotonically increasing with I and II. Then $\eta^* > 1$ will exist for any p^0_{in}, and $\eta(0) = 1$, $\eta = \eta^*$ represent the only two real roots of eqn (15). It is also known that the improper integral in eqn (15) is finite.

THE GALERKIN METHOD†

In order to obtain an approximate but complete solution by a Galerkin's procedure, eqn (8) is first non-dimensionalized by introducing

$$Kt = \tau, \quad K^2 = \phi_1/\rho_0 R_1^2, \quad \phi_2/\phi_1 = C, \quad p^0_{in}/\phi_1 = P, \quad \Omega^2 = \frac{\rho_0 R_1^2 \omega^2}{\phi_1}. \tag{18}$$

Next, for oscillatory motions $\eta(\tau)$ is assumed in the form

$$\eta(\tau) = \frac{1}{a}(1 + b\cos\Omega\tau)^{1/2} \tag{19}$$

which involves three unknown parameters a, b and Ω. The initial condition $\dot\eta(0) = 0$ is trivially satisfied while $\eta(0) = 1$ requires

$$a^2 = 1 + b. \tag{20}$$

Considering $\dot\eta = 0$ at $\eta = \eta^*$, another equation relating a and b is obtained,

$$a^2\eta^{*2} = 1 - b. \tag{21}$$

From eqns (20) and (21), it is clear that $|b| < 1$, $b < 0$.

To determine the remaining parameter, Ω, Galerkin's orthogonalization process is followed. The assumed form of $\eta(\tau)$ is substituted into eqn (8) and the resulting equation is orthogonalized with respect to weighting function $\cos\Omega\tau$ over one complete cycle of oscillations,

$$\int_0^{2\pi} G(z) z \, d(\Omega\tau) = 0, \tag{22}$$

where

$$z = \cos\Omega\tau,$$

and

$$\begin{aligned} G(z) = {} & \frac{2Pa^{2\alpha}}{(1+bz)^\alpha} + \frac{[C(1+S_1) - S_1]a^2}{(1+a^2S_1)+bz} + \frac{4a^2S_1 + m_1S_1b^2\Omega^2(1-z^2)}{4(1+bz)[(1+a^2S_1)+bz]} - \frac{Ca^2(1+S_2)}{(1+a^2S_1)+bz} \\ & + \frac{b^2\Omega^2 m_2(S_2 - S_1)(1-z^2)}{4[(1+a^2S_1)+bz][(1+a^2S_1)+bz]} + (C-1)\ln(1+S_1) - C\ln(1+S_2) + \ln\left(1 + \frac{a^2S_1}{1+bz}\right) \\ & + C\ln\frac{(a^2S_2+1)+bz}{(a^2S_1+1)+bz} + \frac{b\Omega^2}{2a^2}\left[m_1 z \ln\left(1+\frac{a^2S_1}{1+bz}\right) + m_2 z \ln\left(S_2 + \frac{1+bz}{a^2}\right)\right. \\ & \left. - m_2 \ln\left(S_1 + \frac{1+bz}{a^2}\right)\right]. \end{aligned} \tag{23}$$

The integral defined by eqn (22) is evaluated by contour integration for $\alpha = 1$. The frequency of oscillations for $m_1 = m_2 = 1$ is given by

$$\Omega^2 = \frac{4a^2}{b^2} \frac{H}{\ln\dfrac{B + \sqrt{(B^2 - b^2)}}{1 + \sqrt{(1-b^2}}} \tag{24}$$

†The details of this process are omitted here since they are essentially the same as those given in Ref. [6].

where

$$H=\sqrt{(A^2-b^2)}-A^2-a^2(1+S_1)\frac{A}{\sqrt{(A^2-b^2)}}+\frac{(1+2P)a^2+b^2-1}{\sqrt{(1-b^2)}}+a^2(S_1-2P)+1$$
$$+C\left[\frac{B^2-b^2-a^2B(S_2+1)}{\sqrt{(B^2-b^2)}}+\frac{a^2(S_1+1)A-A^2+b^2}{\sqrt{(A^2-b^2)}}\right], \quad (25)$$
$$A=1+a^2S_1, \quad B=1+a^2S_2.$$

For small amplitude oscillations, letting $a\to 1$, $b\to 0$, $P\to 0$, and applying L'Hospital's rule in eqn (24) a frequency expression identical with eqn (13) is obtained.

ILLUSTRATIVE EXAMPLE AND DISCUSSION OF THE RESULTS

To provide some numerical results, a tube of two layers with $R_2/R_1=2$, $R_3/R_1=3$, $\phi_2/\phi_1=2$ and $\alpha=1$, 1.2, 1.4, 1.6 is considered and the frequencies of oscillations corresponding to different initial pressures are determined by exact formulation. The results for $\alpha=1$ are compared with those obtained by the approximate formulation.

The integral $\int_1^{\eta^*}\dot{\eta}^{-1}\,d\eta$ does not seem to have a closed form solution. This integral is evaluated numerically by Newton–Cotes integration formula (see, for example, Ref. [9], formula 25, 4.40, p. 887). The interval of integration $(1,\eta^*)$ is first divided into sub-intervals $(1,1+\delta)$, $(1+\delta,\eta^*-\delta)$, $(\eta^*-\delta,\eta^*)$. The inner interval is further divided into 1000 segments. The integrals in the intervals $(1,1+\delta)$ and $(\eta^*-\delta,\eta^*)$ are approximated by integrals in the intervals $(1+\epsilon,1+\delta)$ and $(\eta^*-\delta,\eta^*-\epsilon)$ and these intervals are further divided into 500 segments. δ is chosen to be 10^{-4} and the smallest ϵ not to cause numerical sensitivity on an IBM 370-145 machine is 0.5×10^{-7}. A good convergence is obtained at these values as it can be observed from Figs. 1 and 2.

The maximum relative amplitude η^* of the inner radius is plotted in Fig. 3 as a function of the non-dimensionalized initial pressure $P=p_{in}^0/\Phi_1$ for $\alpha=1$, 1.2, 1.4, 1.6. These curves display a hardening behavior with increasing initial pressure.

Figure 4 shows the non-dimensionalized frequency $\bar{\omega}/\bar{\omega}_0$ ($\bar{\omega}^2=\rho_0R_1{}^2\omega^2/\phi_1$, ω_0 = frequency of small oscillations) plotted against the initial pressure P. A hardening behavior is observed for $\alpha=1.2$, 1.4, 1.6 at relatively low values of P while the tube acts as a soft spring at large P. For $\alpha=1$ a softening behavior is observed for all P. This behavior is more pronounced for small P.

For the same numerical data, the approximate expression for the frequency of oscillations given by eqn (24) is evaluated in closed form by contour integration for $\alpha=1$. The isolated points in Fig. 4 represent the results obtained by Galerkin's method. An inspection of Table 1

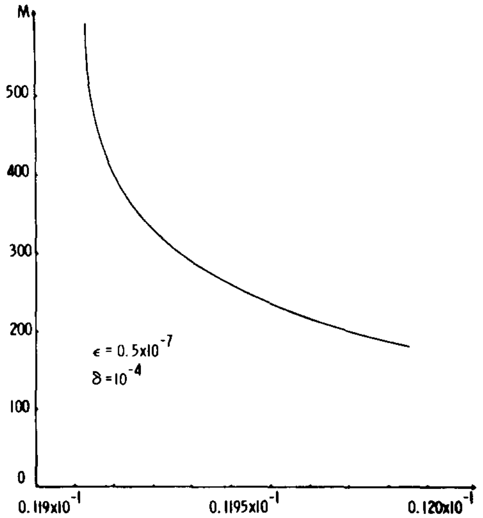

Fig. 1. Number of segments vs $\int_{1+\epsilon}^{1+\delta} d\eta/\dot{\eta}$.

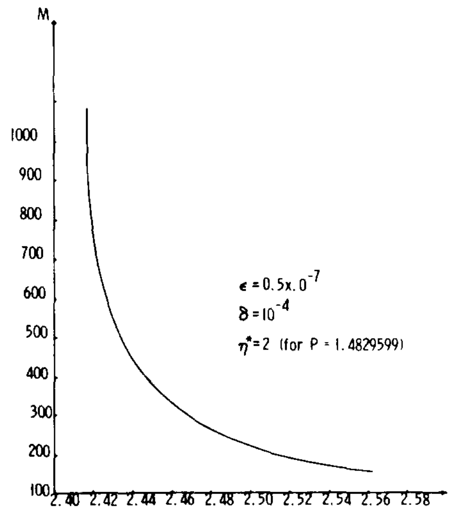

Fig. 2. Number of segments vs $\int_{1+\delta}^{\eta^*-\delta} d\eta/\dot{\eta}$.

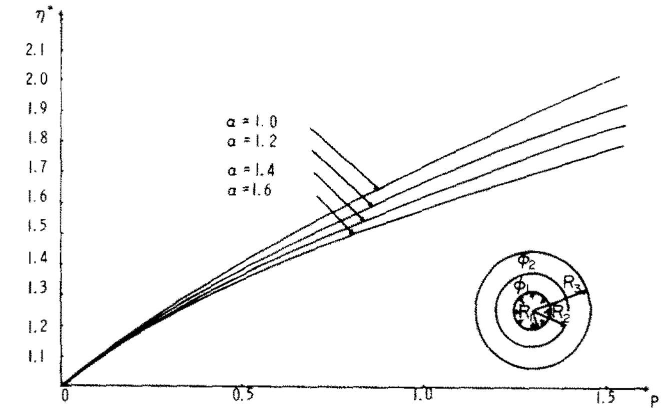

Fig. 3. Maximum amplitude vs P.

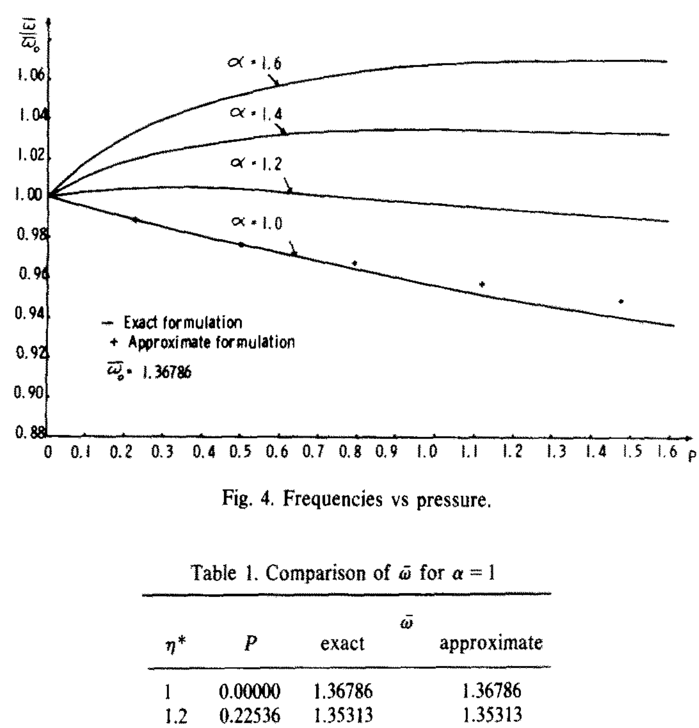

Fig. 4. Frequencies vs pressure.

Table 1. Comparison of $\bar{\omega}$ for $\alpha = 1$

η^*	P	$\bar{\omega}$ exact	$\bar{\omega}$ approximate
1	0.00000	1.36786	1.36786
1.2	0.22536	1.35313	1.35313
1.4	0.48885	1.33637	1.33888
1.6	0.78797	1.31937	1.32500
1.8	1.12017	1.31091	1.31145
2.0	1.48296	1.29416	1.29829
2.2	1.87405	1.27013	1.28554
2.4	2.29140	1.26247	1.27325

and Fig. 4 indicate an excellent agreement between the results obtained by exact and approximate formulations. The approximate method avoids the singular points and the computation time is significantly reduced (0.12 sec as compared to 11 sec by the exact formulation to compute a frequency).

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